Functions

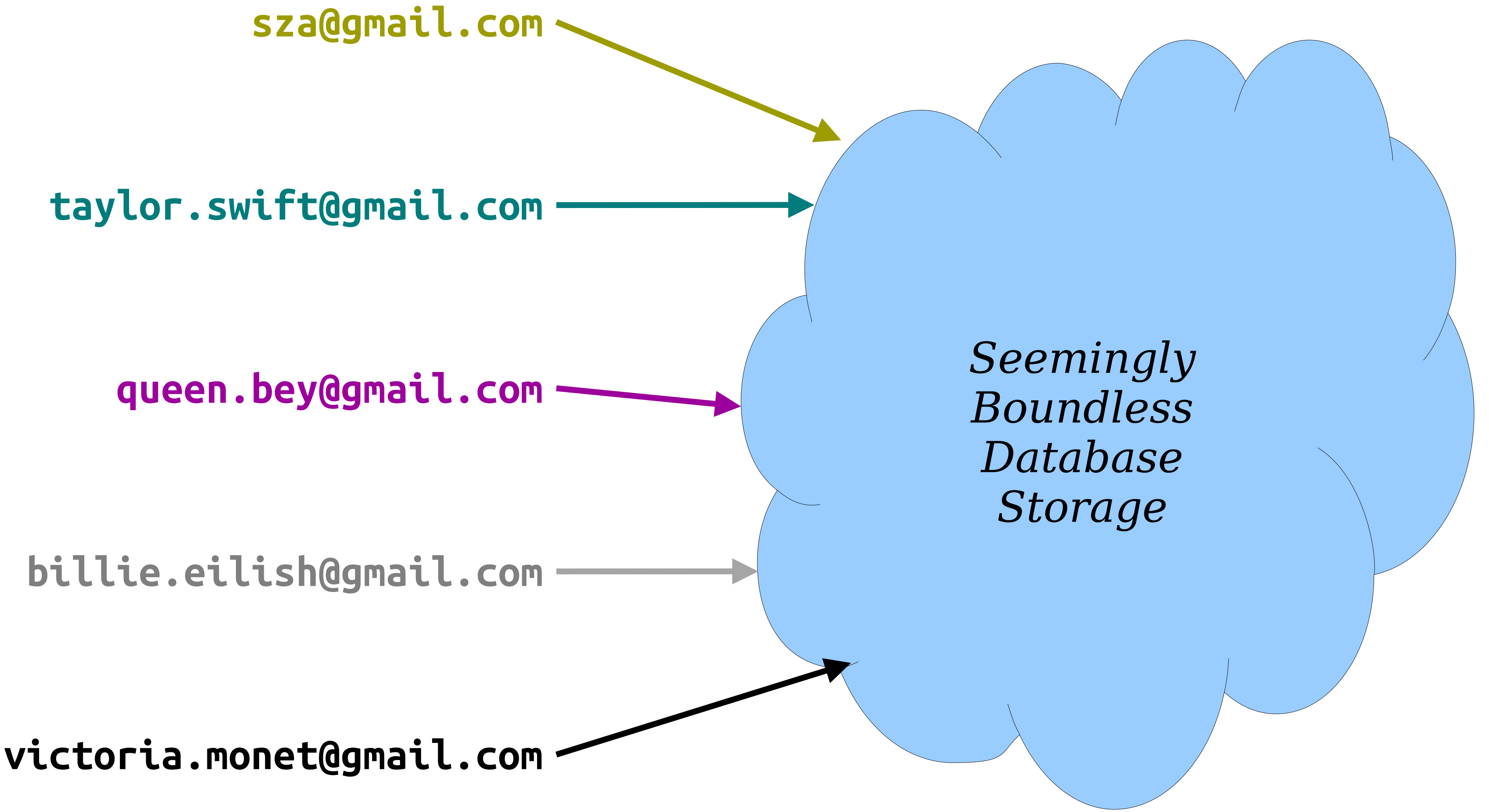
Part One

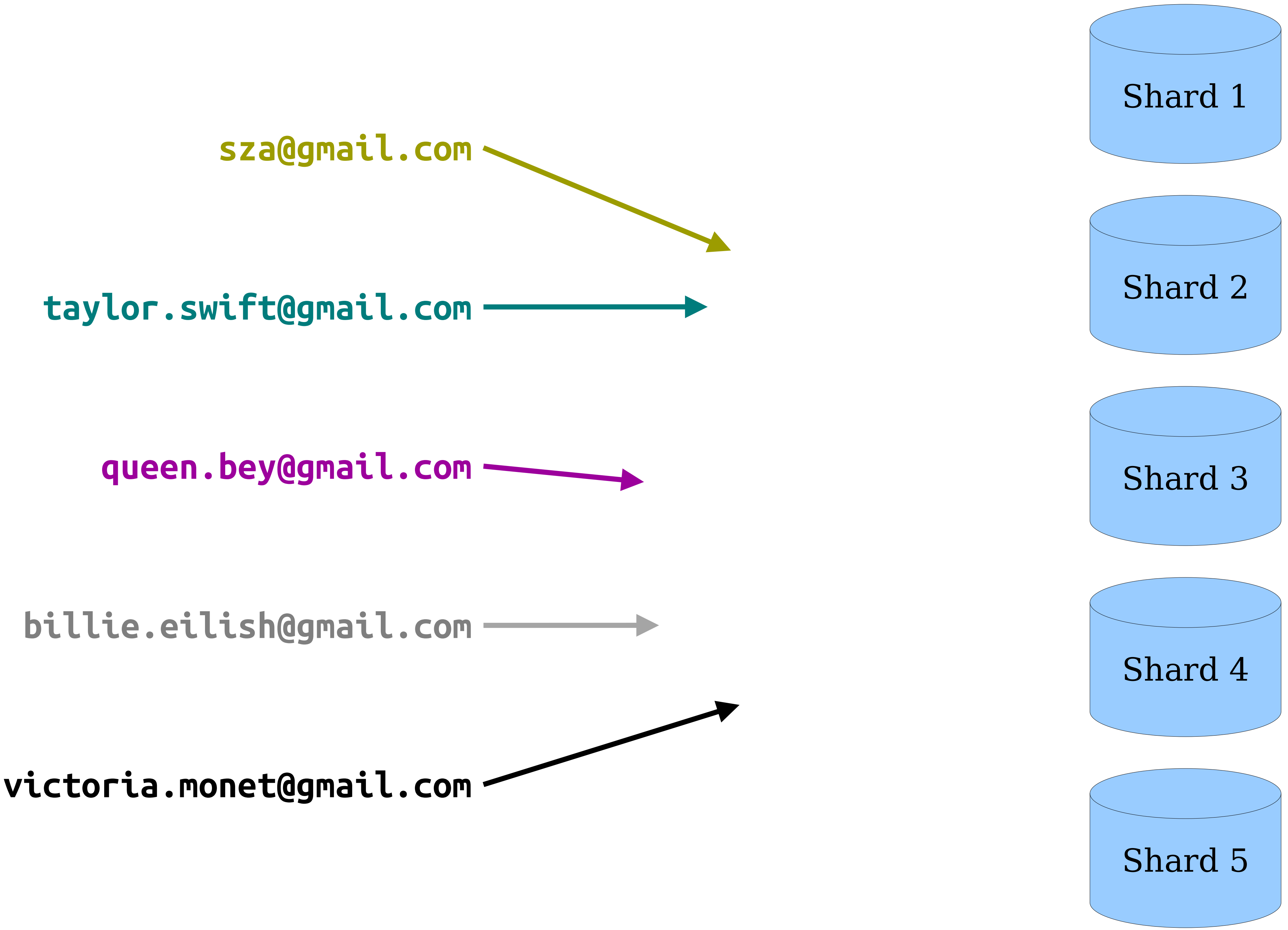
Outline for Today

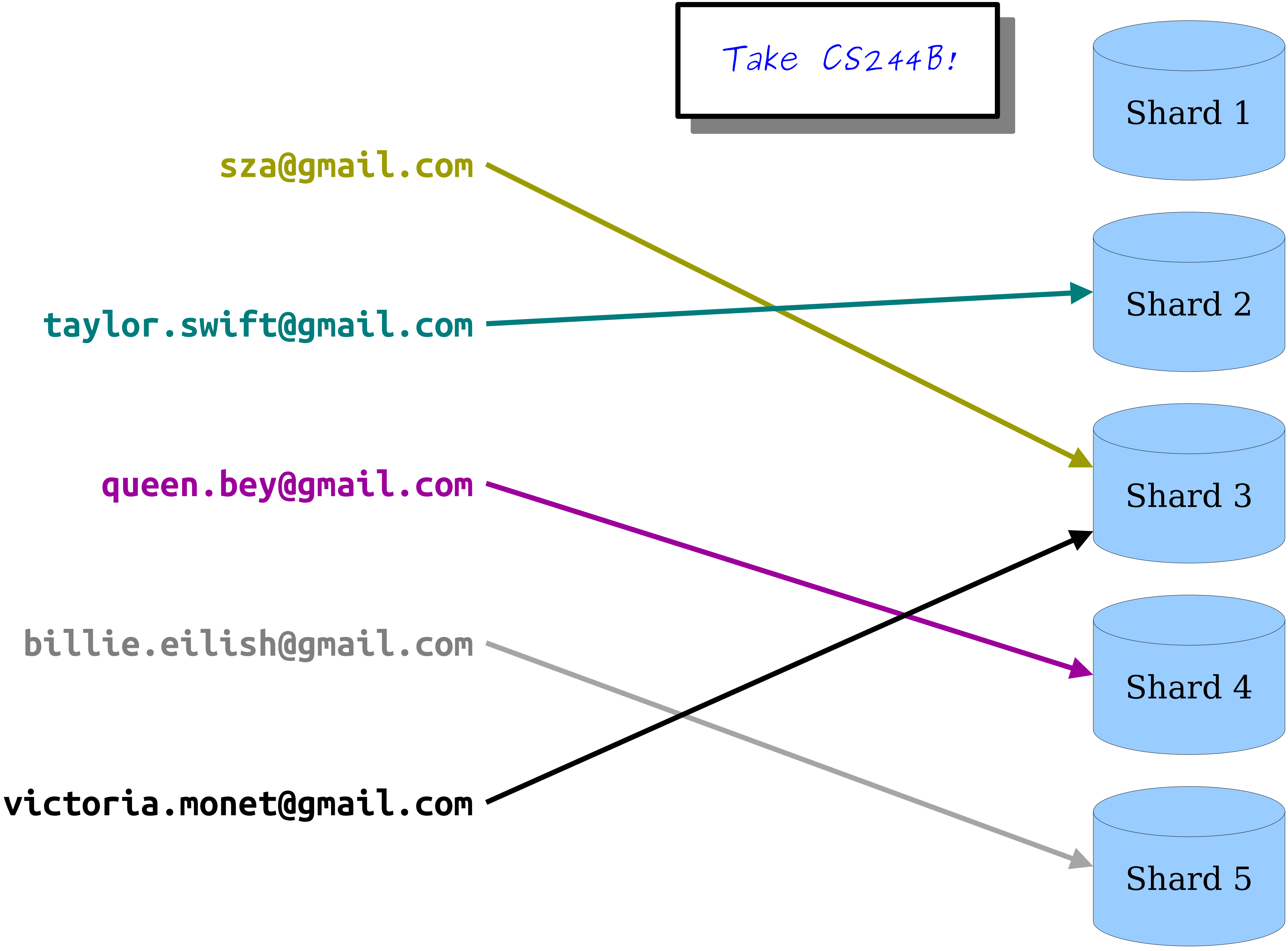
- ***What is a Function?***
  - It's more nuanced than you might expect.
- ***Domains and Codomains***
  - Where functions start, and where functions end.
- ***Defining a Function***
  - Expressing transformations compactly.
- ***Special Classes of Functions***
  - Useful types of functions you'll encounter IRL.
- ***Proofs on First-Order Definitions***
  - A key skill.

What is a function?

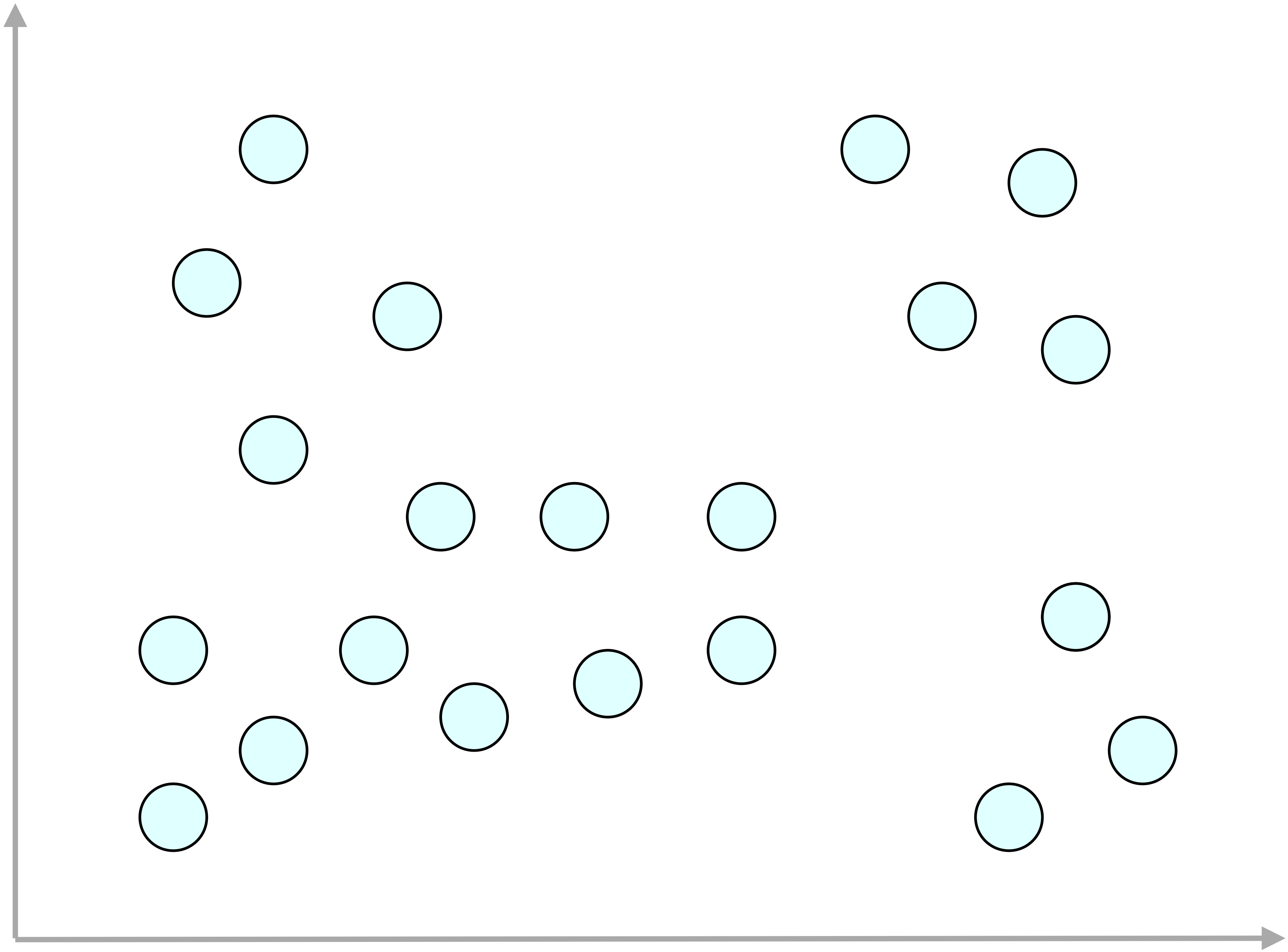
Motivating Example 1: Database Sharding

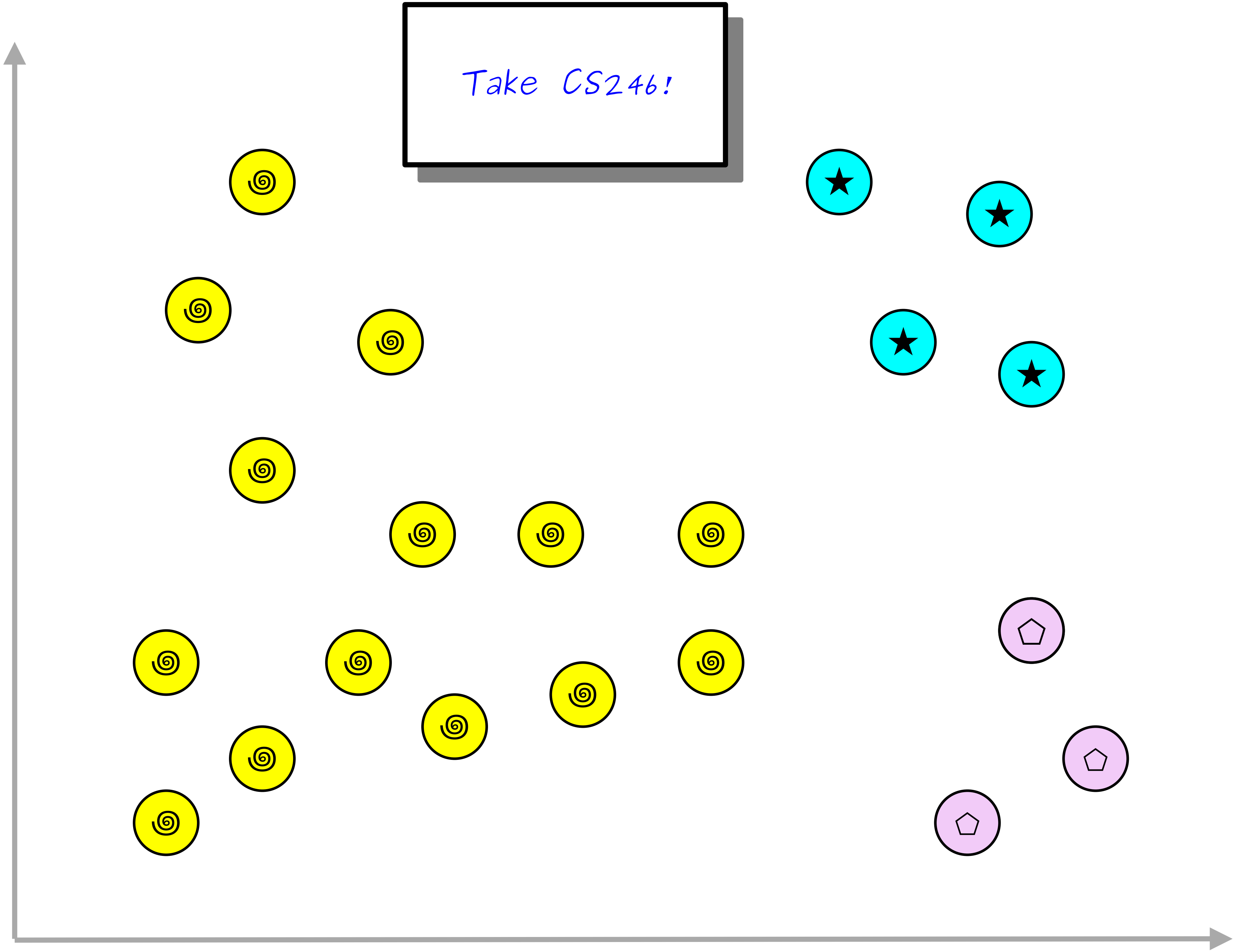

sza@gmail.com
taylor.swift@gmail.com
queen.bey@gmail.com
billie.eilish@gmail.com
victoria.monet@gmail.com
Seemingly
Boundless
Database
Storage


sza@gmail.com
taylor.swift@gmail.com
queen.bey@gmail.com
billie.eilish@gmail.com
victoria.monet@gmail.com
Shard 1
Shard 2
Shard 3
Shard 4
Shard 5


Take CS244B!
sza@gmail.com
taylor.swift@gmail.com
queen.bey@gmail.com
billie.eilish@gmail.com
victoria.monet@gmail.com
Shard 1
Shard 2
Shard 3
Shard 4
Shard 5

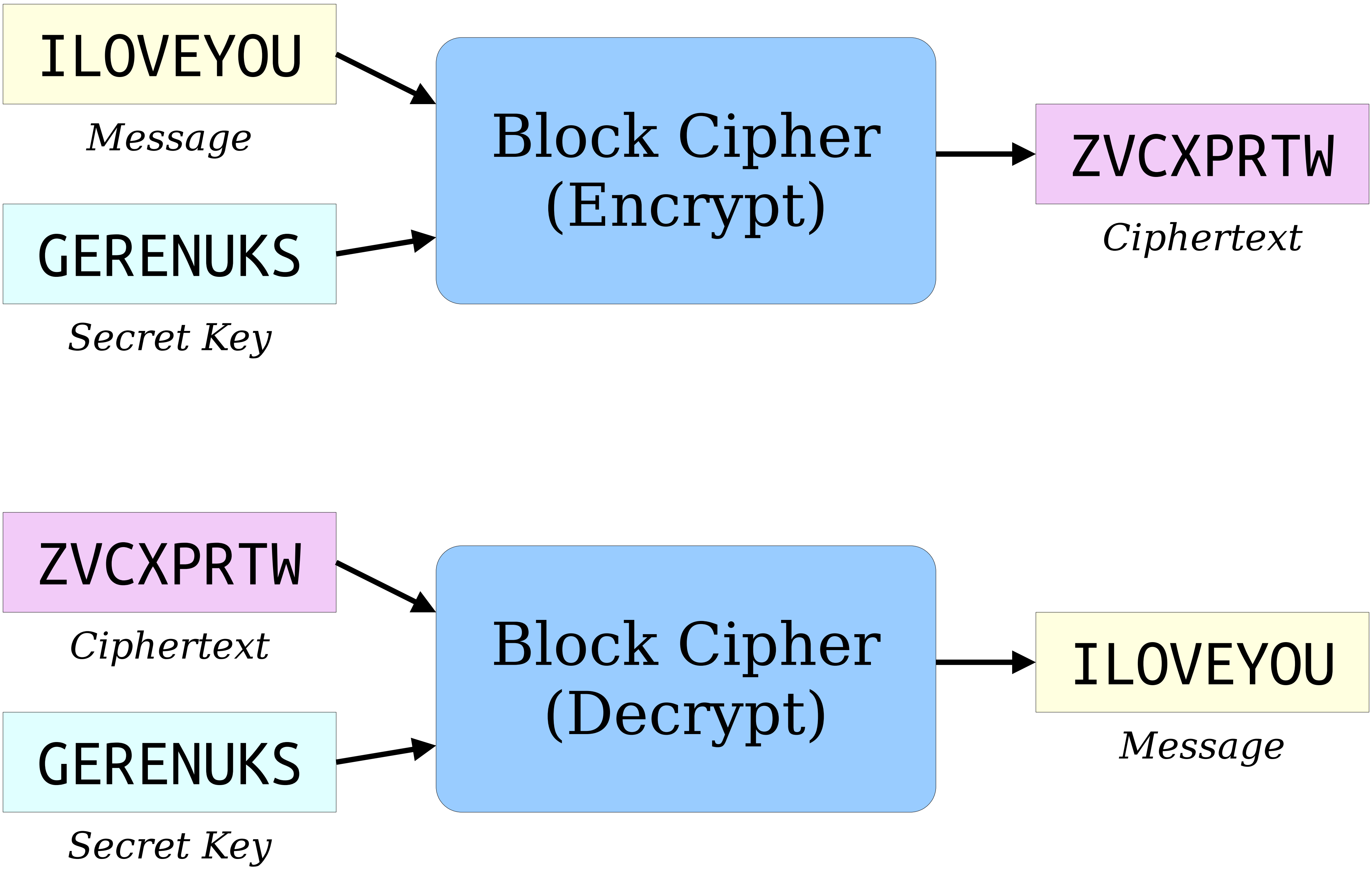
Motivating Example 2: Data Clustering




Take CS246!

Motivating Example 3: Block Ciphers

Take CS255!

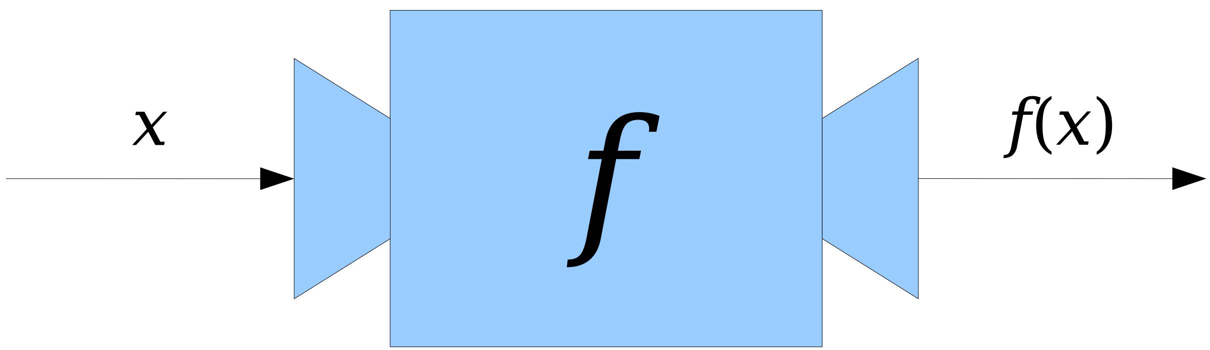

ILOVEYOU
Message
GERENUKS
Secret Key
Block Cipher
(Encrypt)
ZVCXPRTW
Ciphertext
ZVCXPRTW
Ciphertext
GERENUKS
Secret Key
Block Cipher
(Decrypt)
ILOVEYOU
Message

What's In Common?

- We have a fixed, known set of possible inputs.
  - In our examples: user names, 2D data points, and length-8 strings.
- We have a fixed, known set of possible outputs.
  - In our examples: database shards, cluster labels, length-8 strings.
- Each input is assigned an output.
  - Some outputs might be assigned multiple inputs.
  - Some outputs might be assigned no inputs.

High-Level Intuition:

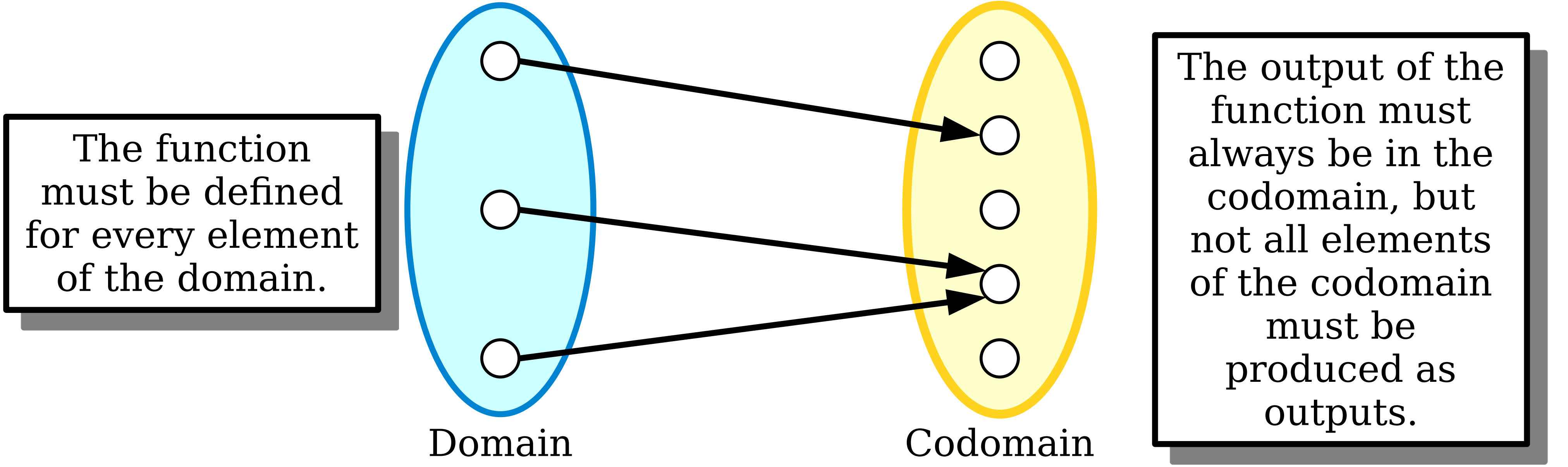
A function is an object *f* that takes in exactly one input *x* and produces exactly one output *f*(*x*).



(This is not definition. It's just to help you build and intuition.)

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.



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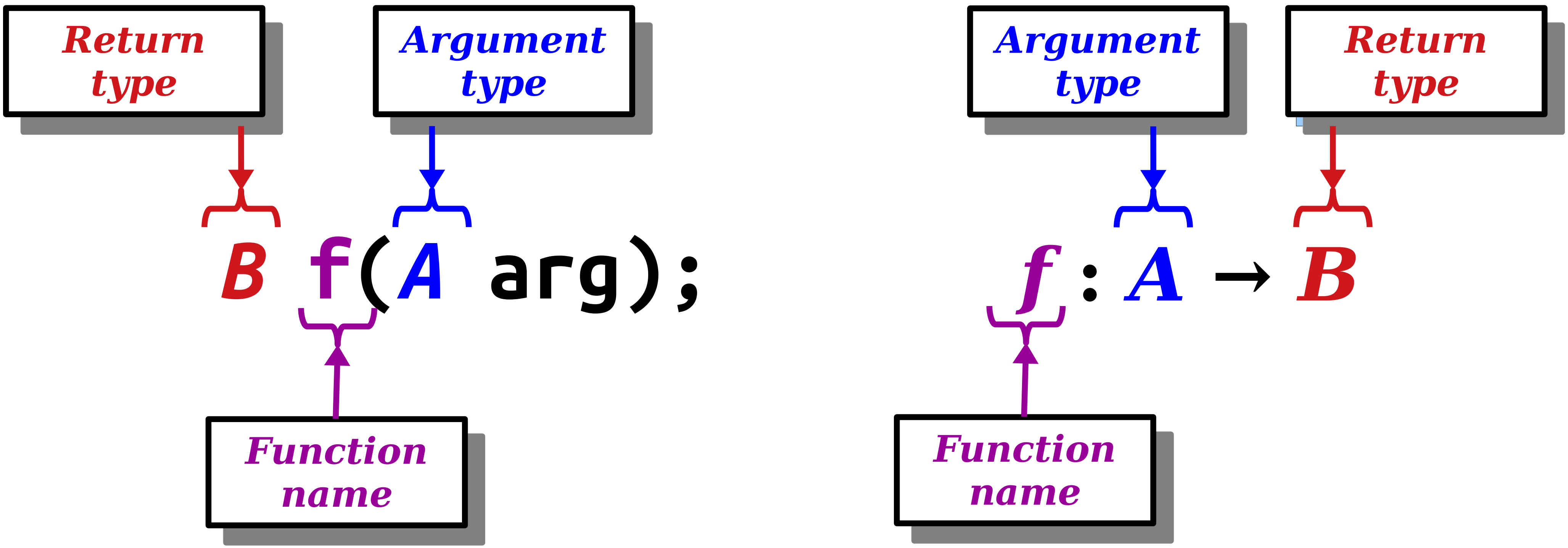
The ***domain*** of this function is ℝ. Any real number can be provided as input.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- Think of this like a "function prototype" in C++.



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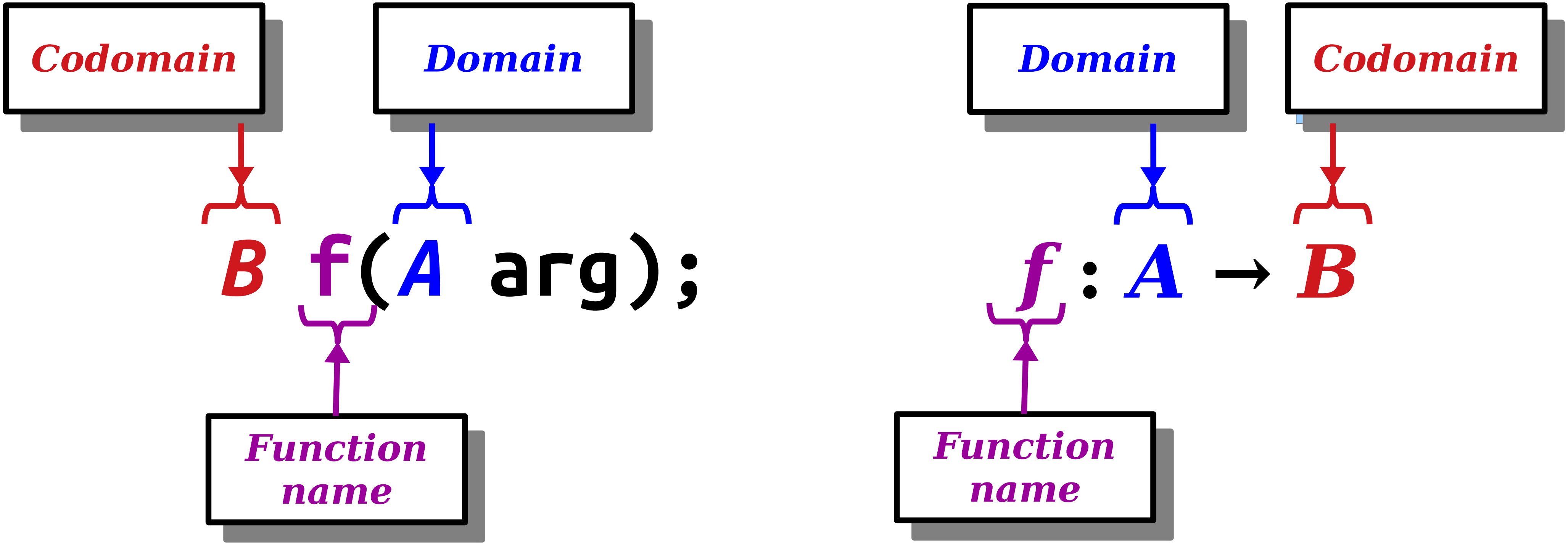


Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
  - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
  - If you're clever, you can get two or more arguments to a function while still obeying this rule. Chat with me after class to learn more!
- In mathematics, functions are ***deterministic*** and can't behave randomly.
  - If you're clever, you can get functions that kinda sorta ish look random. Chat with me after class to learn more!

The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

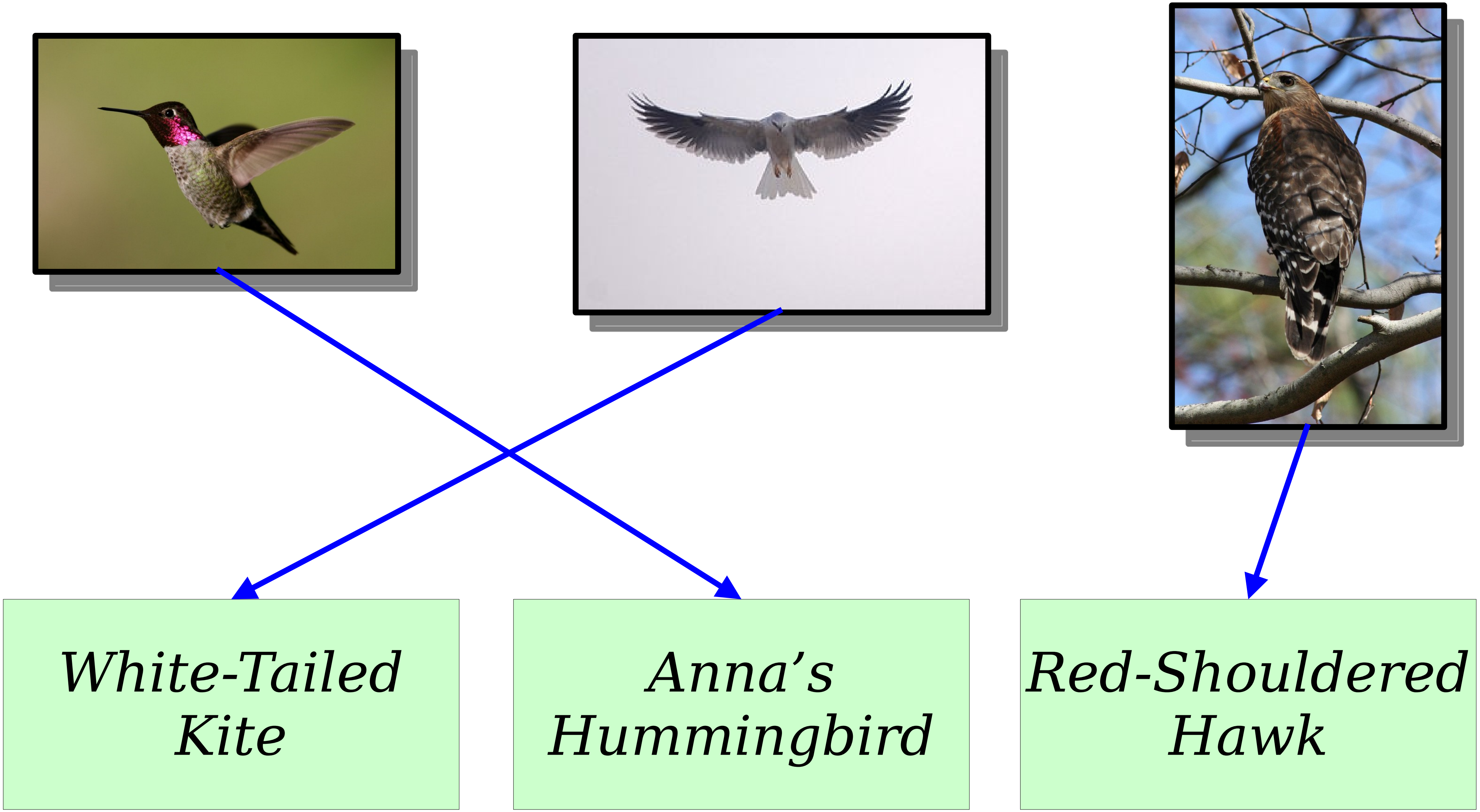
(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

Defining Functions

Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let’s go over a few examples.



Functions can be defined as a ***picture***.
Draw the domain and codomain explicitly.
Then, add arrows to show the outputs.

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where}$$

$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a ***rule***.
Be sure to explicitly state what the domain and codomain are!

$$f : \mathbb{Z} \to \mathbb{N}, \text{ where}$$

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

$$f(x) = \frac{x+2}{x+1}$$

Answer at

https://cs103.stanford.edu/pollev

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

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Answer at

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This expression isn't defined when x = -1, so f isn't defined over its full domain. We therefore don't consider it to be a function.

Is this a function from ℝ to ℝ?

$$f(x) = \frac{x+2}{x+1}$$

Answer at

https://cs103.stanford.edu/pollev

Is this a function from $\mathbb{N}$ to $\mathbb{R}$?

$$f(x) = \frac{x+2}{x+1}$$

Answer at

https://cs103.stanford.edu/pollev

Yep, it's a function! Every natural number maps to some real number.

Is this a function from ℕ to ℝ?

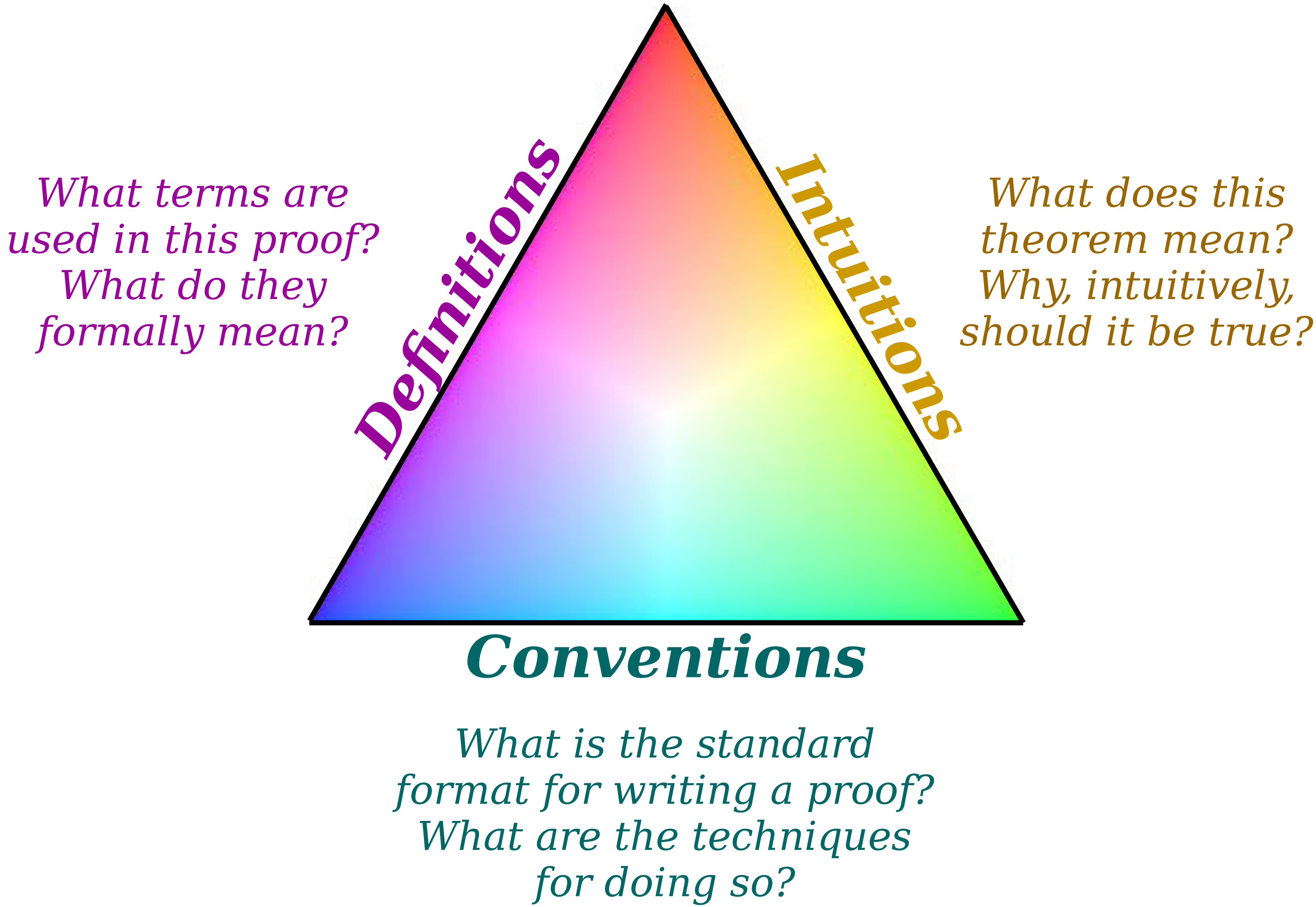
Time-Out for Announcements!

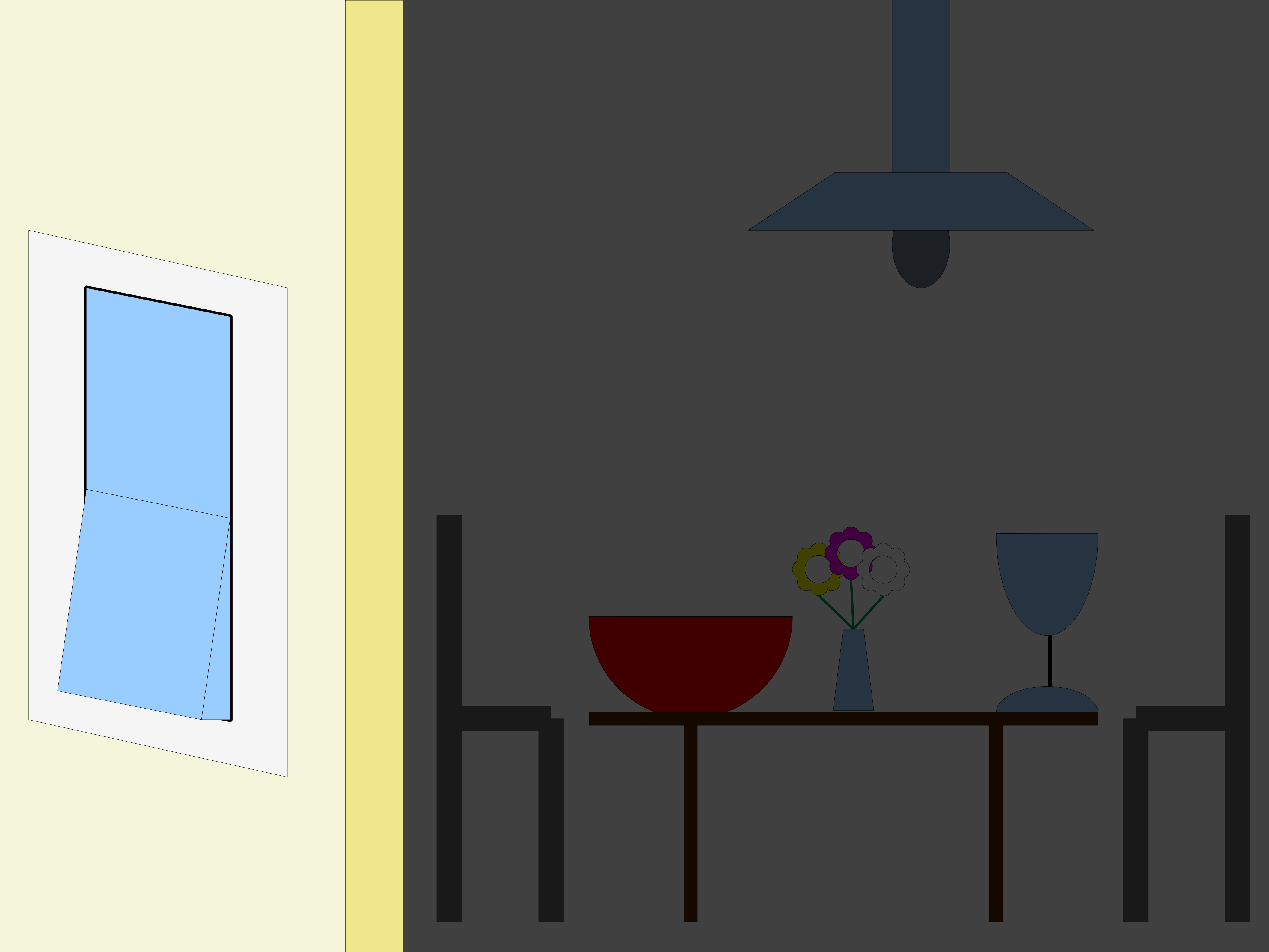
Problem Set One Solutions

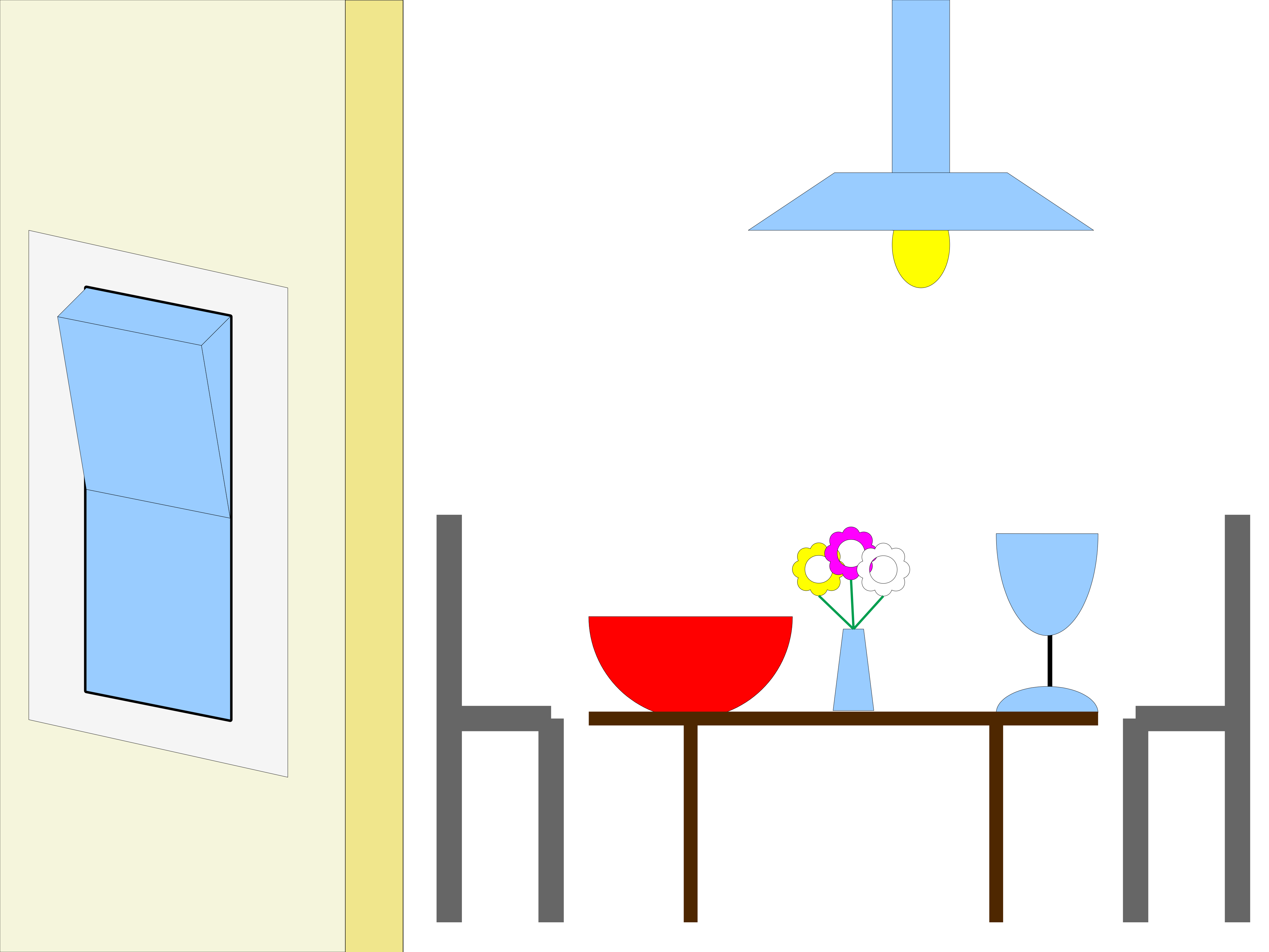
- We've just posted solutions to Problem Set One. They're linked from the main PS1 page.
- We recommend you read over our solution set before finishing PS2.
  - You'll get to see examples of polished written proofs.
  - Each problem has a "Why We Asked This Question" section, which gives some context.
  - We may have solved the problem differently than you, and this will give you more perspectives to use.
- We'll aim to have PS1 graded and returned tomorrow in the late afternoon / evening.

Back to CS103!

Special Types of Functions


What terms are used in this proof? What do they formally mean?
Definitions
Intuitions
What does this theorem mean? Why, intuitively, should it be true?
Conventions
What is the standard format for writing a proof? What are the techniques for doing so?





Undoing by Doing Again

- Some operations invert themselves. For example:
  - Flipping a switch twice is the same as not flipping it at all.
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Building encryption systems (symmetric block ciphers).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

This is the formal definition. Use it in proofs.

This is just an intuition. Don't use it in proofs.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $g : \mathbb{Z} \to \mathbb{Z}$ defined as $g(x) = -x$.
  - $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = {}^1/_x$.
  - $p : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

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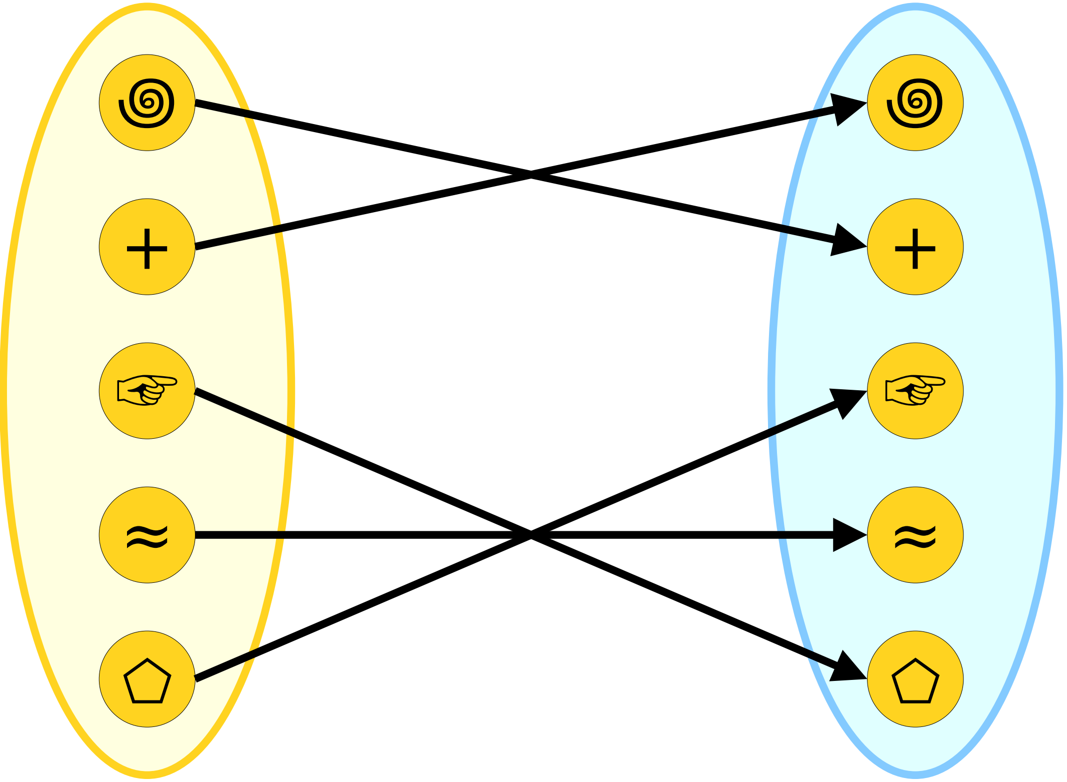
Involutions, Visually



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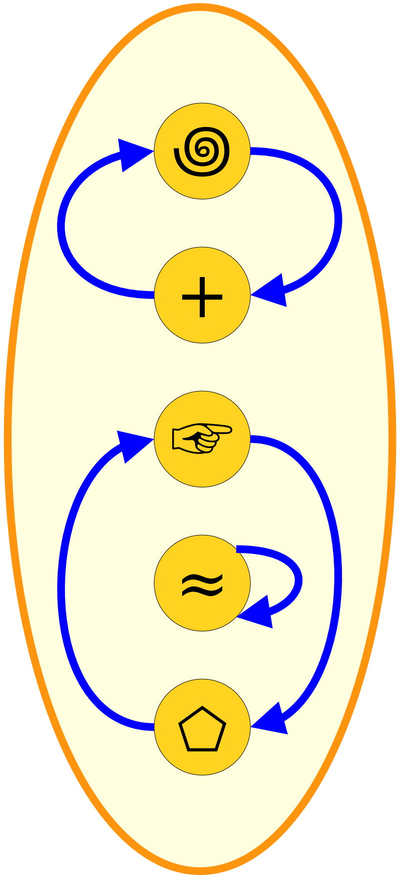

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Proofs on Involutions

Theorem: The function $f : \mathbb{Z} \to \mathbb{Z}$ defined as

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Proof: Pick some $n \in \mathbb{Z}$. We need to show that $f(f(n)) = n$. To do so, we consider two cases.

Case 1: n is even.

Case 2: n is odd.

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Case 2: n is odd.

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$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg\forall n \in \mathbb{N}.\ f(f(n)) = n$$
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Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

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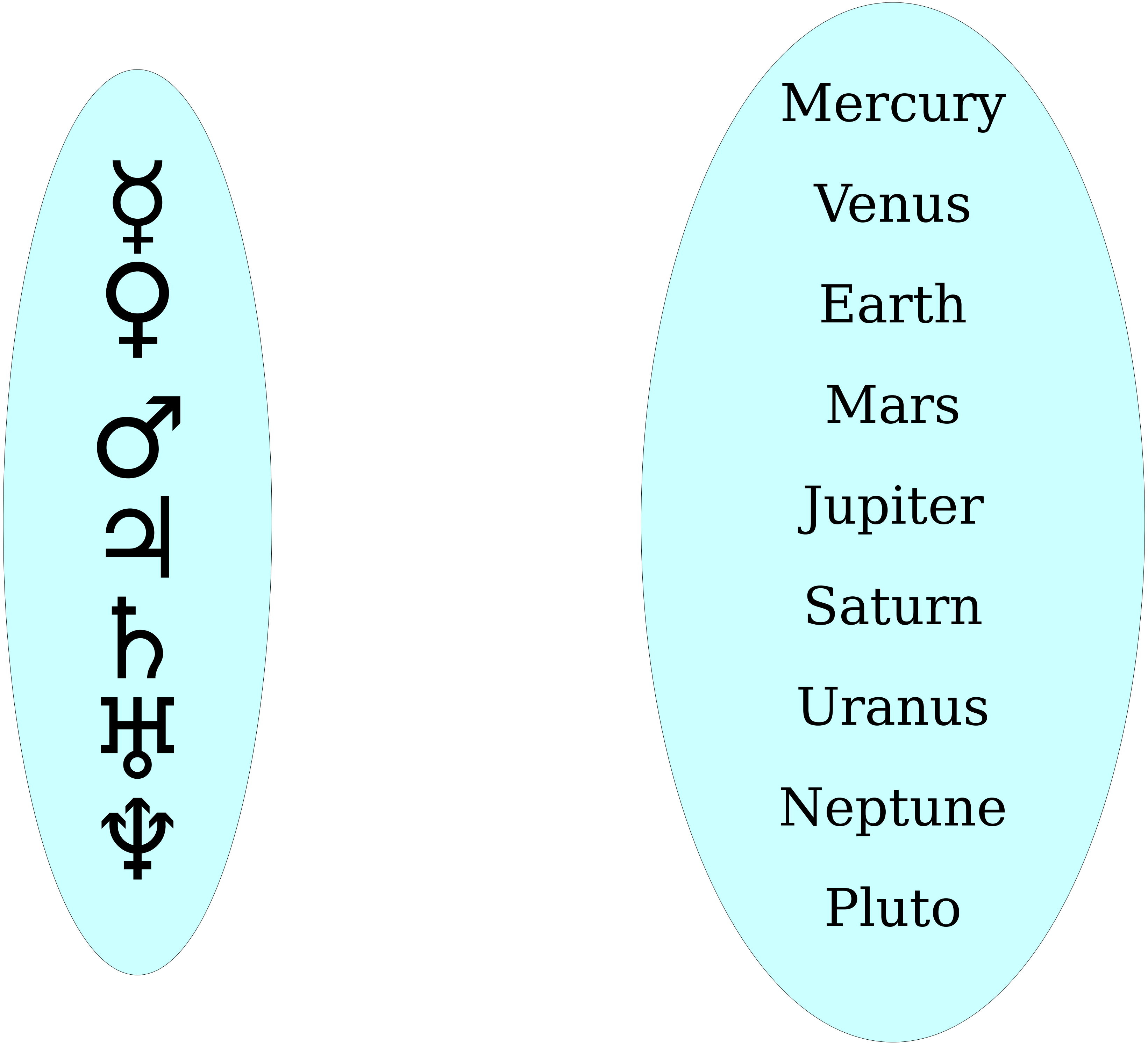
Another Class of Functions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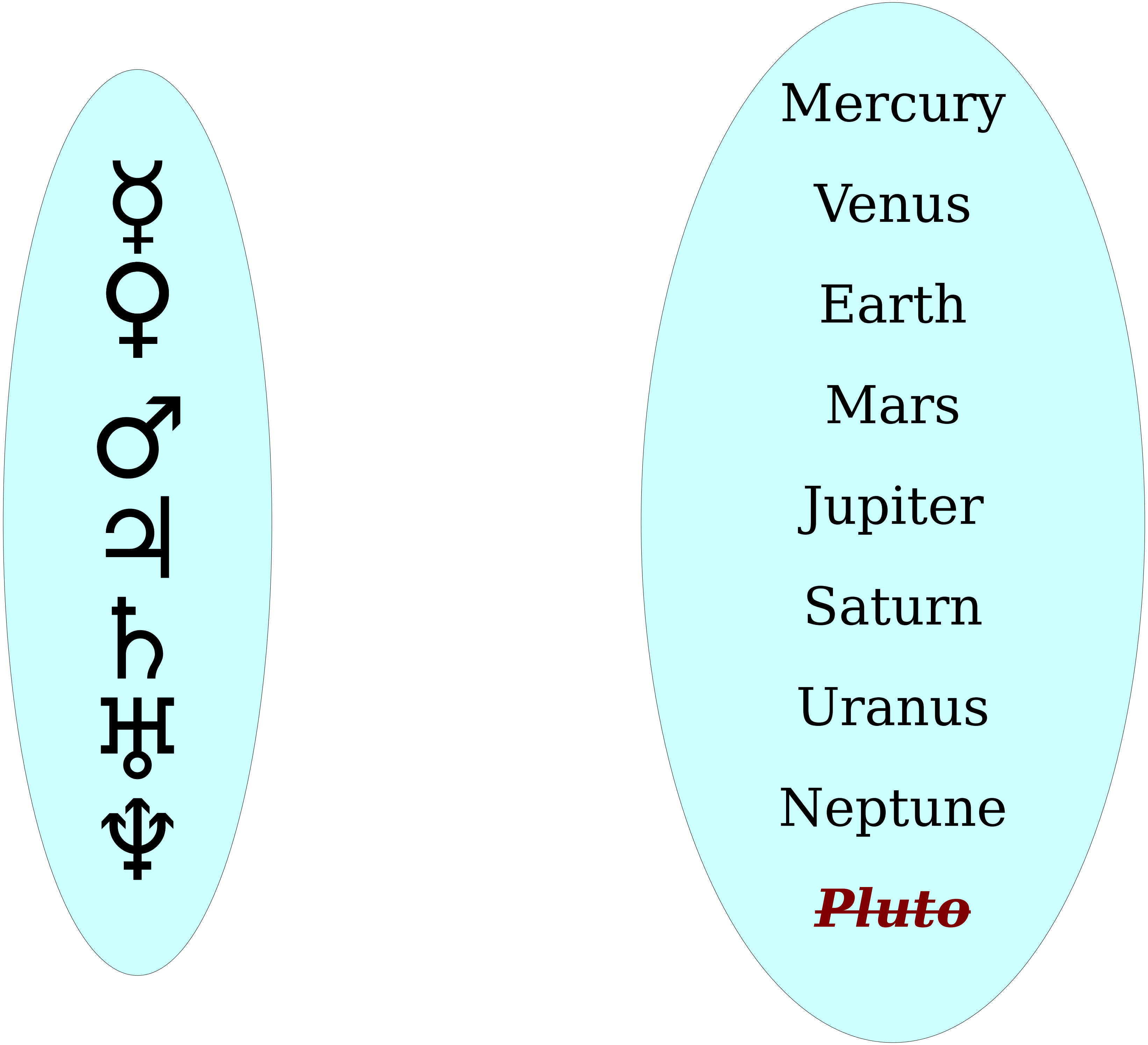
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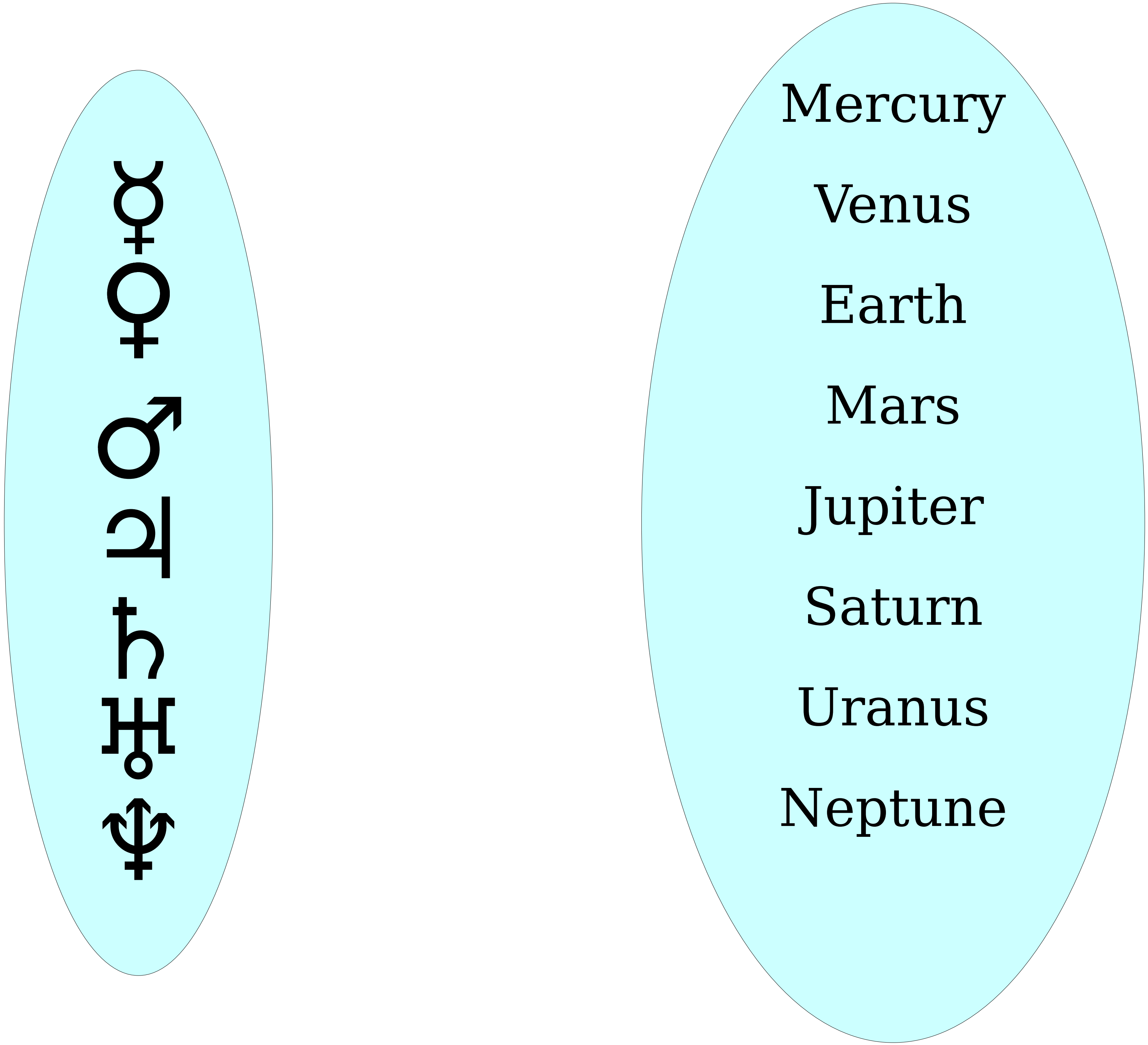
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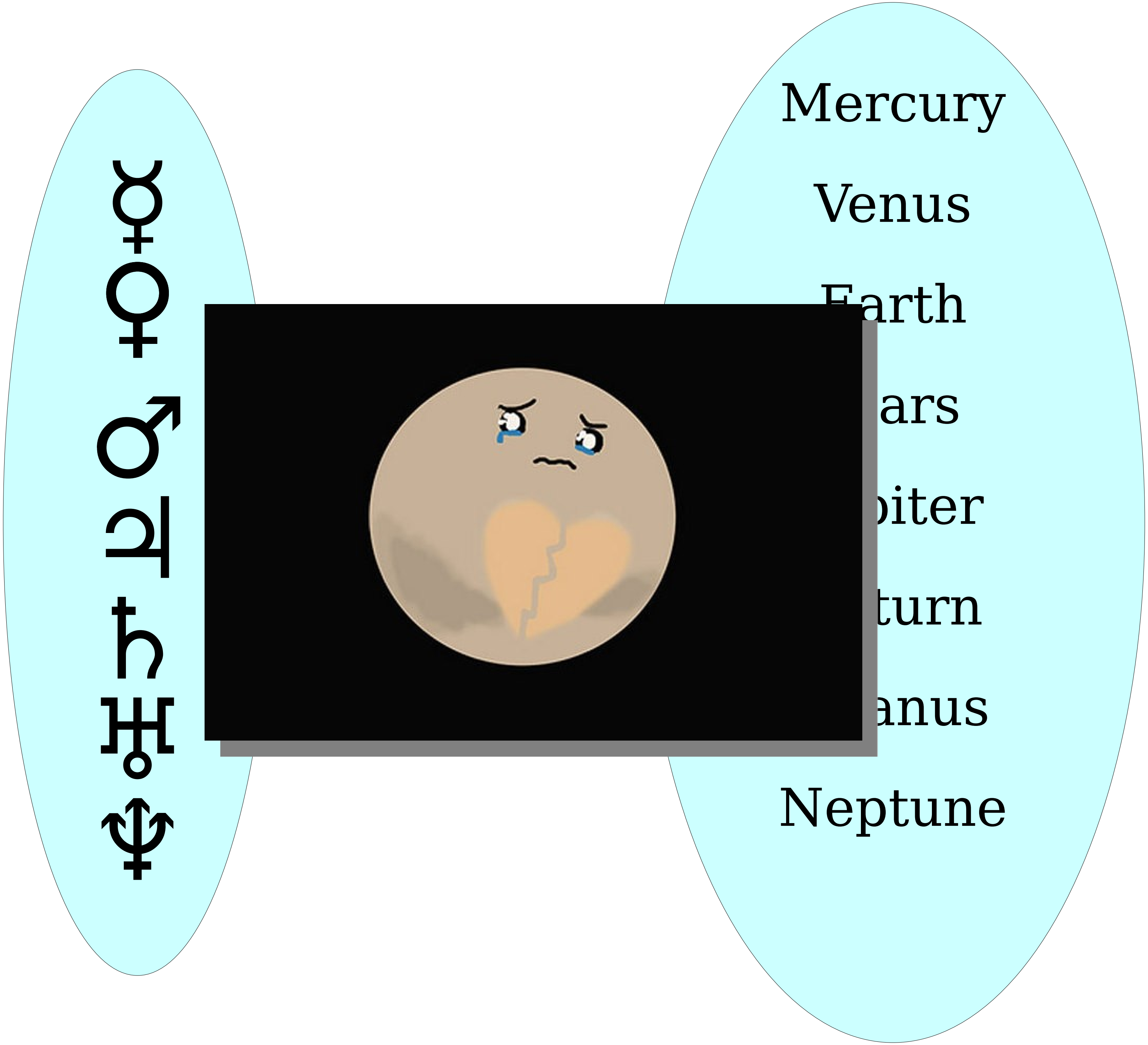
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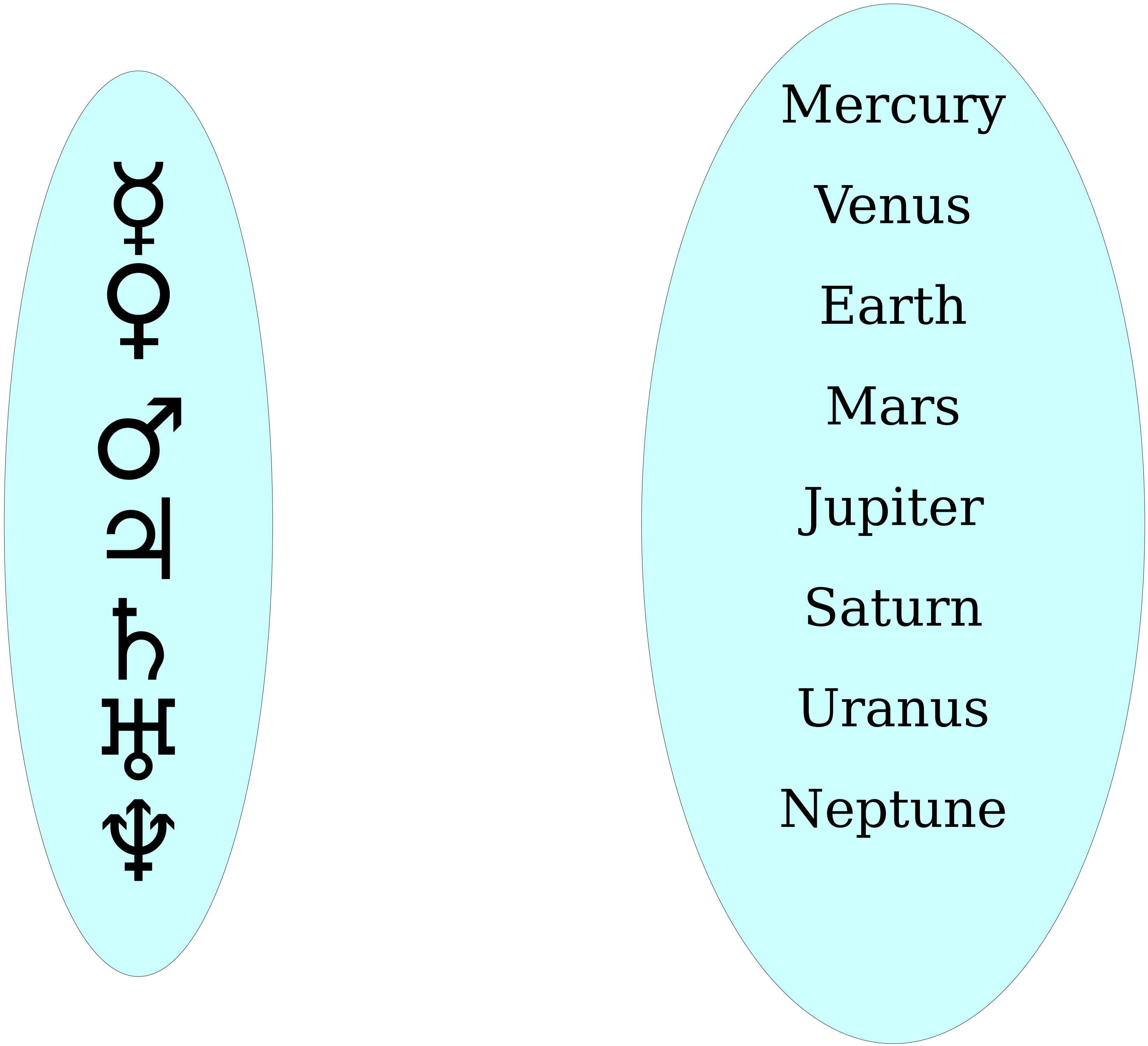
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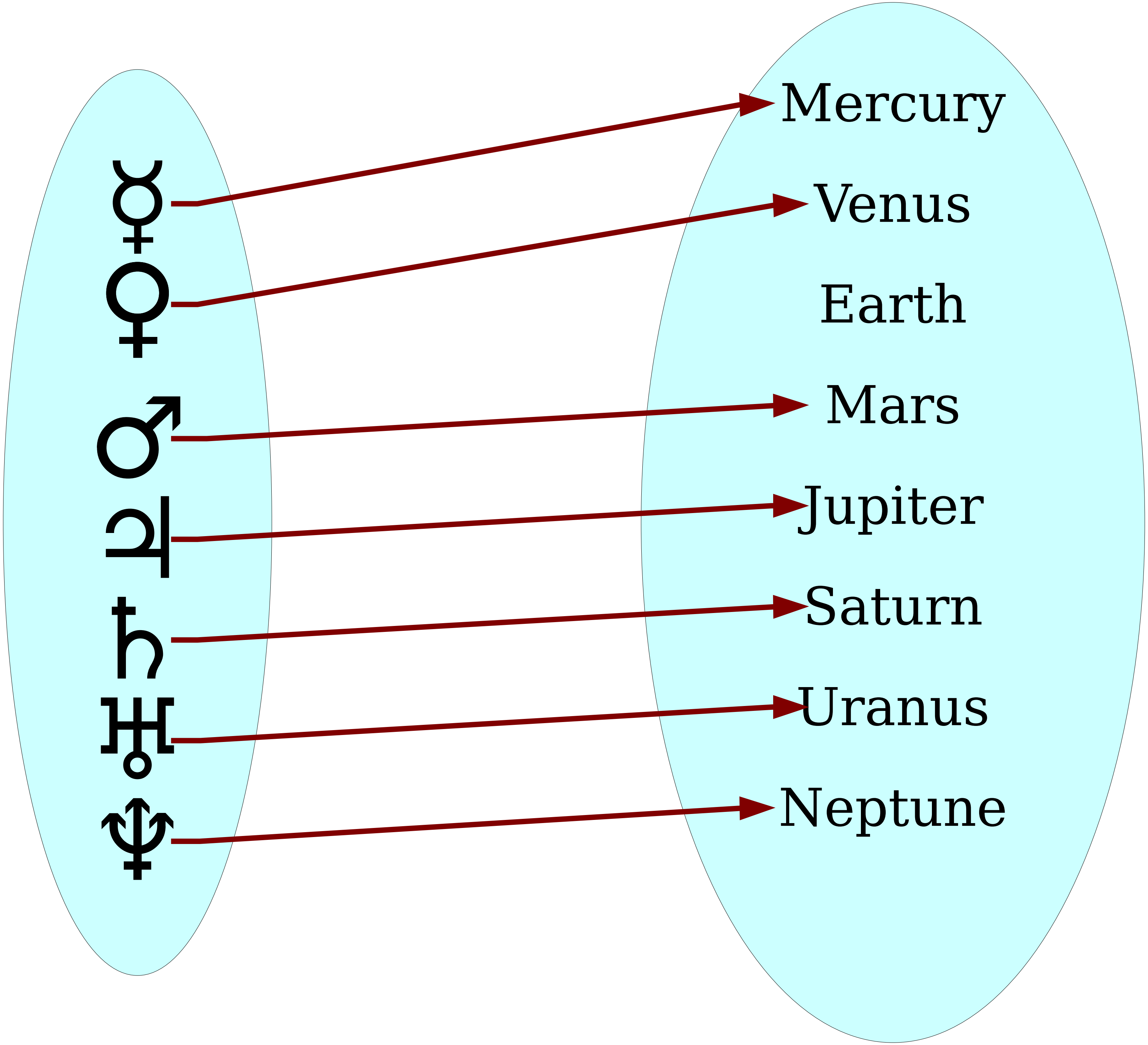
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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injections

- Let be the set of all CS103 students. Which of the following are injective?
  - f : → ℕ where $f(x)$ is x's Stanford ID number.
  - g : → , where is the set of all continents and $g(x)$ is x's continent of birth.
  - h : → , where is the set of all given (first) names, where $h(x)$ is x's given (first) name.

Answer at

https://cs103.stanford.edu/pollev

$f : A \to B$ is ***injective*** when either equivalent statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
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Proofs on Injections

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Good exercise: Repeat this proof using the other definition of injectivity!

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$A \to B$		Assume A is true, then prove B is true.
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What is the negation of this statement?

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$$\exists x_1 \in \mathbb{Z}. \exists x_2 \in \mathbb{Z}. (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.
Can we do that?

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What does it mean for f to be injective?

$$\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

What is the negation of this statement?

$$\neg\forall x_1 \in \mathbb{Z}.\ \forall x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$
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$$\exists x_1 \in \mathbb{Z}.\ \exists x_2 \in \mathbb{Z}.\ (x_1 \neq x_2 \land f(x_1) = f(x_2))$$

Therefore, we need to find $x_1, x_2 \in \mathbb{Z}$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$. Can we do that?

Injective Functions

Theorem: Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as $f(x) = x^4$. Then f is not injective.

Proof:

What does it mean for f to be injective?

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Let $x_1 = -1$ and $x_2 = +1$.

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Let $x_1 = -1$ and $x_2 = +1$. Notice that

$$f(x_1) = f(-1) = (-1)^4 = 1$$

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and

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and

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so $f(x_1) = f(x_2)$ even though $x_1 \neq x_2$, as required.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

		To ***prove*** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove that A is true for that specific choice of x.
$A \to B$		Assume A is true, then prove B is true.
$\neg A$		Simplify the negation, then consult this table on the result.

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$A \land B$		Prove A. Also prove B.
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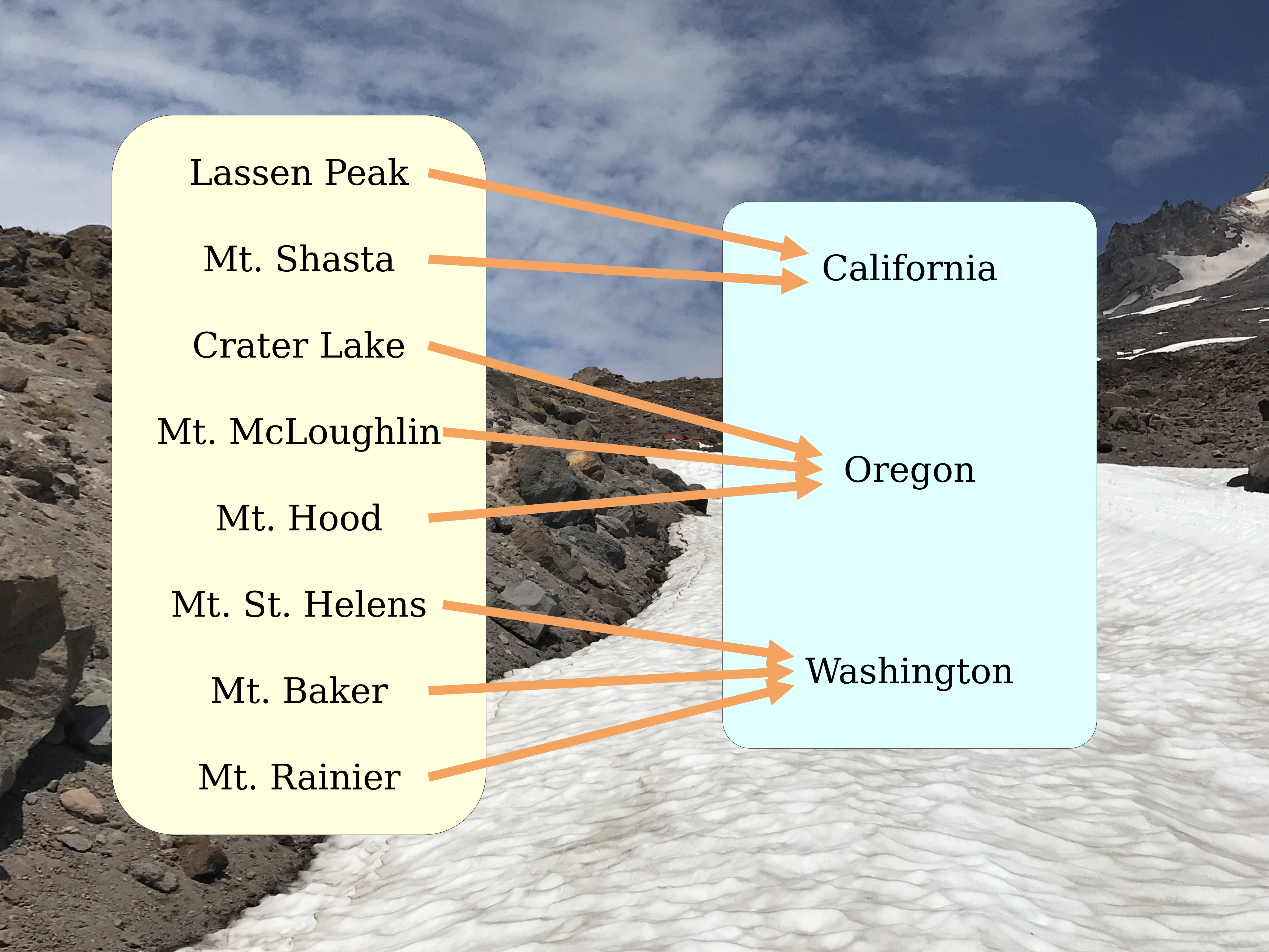
$A \to B$		Assume A is true, then prove B is true.
$A \land B$		Prove A. Also prove B.
$A \lor B$		Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*
$A \leftrightarrow B$		Prove $A \to B$ and $B \to A$.
$\neg A$		Simplify the negation, then consult this table on the result.

Two More Classes of Functions


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's an input that produces it."*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

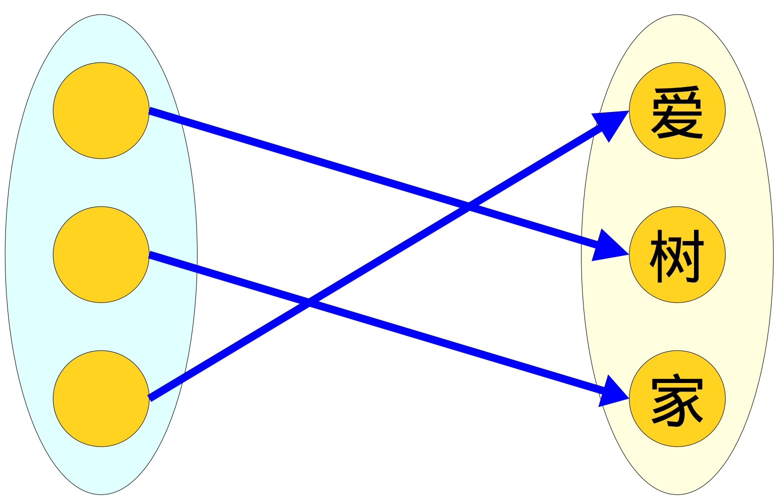
Check the appendix for sample proofs involving injections.

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \to B$ is a bijection, then f represents a way of pairing off elements of A and elements of B.



Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 1$.
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Next Time

- ***First-Order Assumptions***
  - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.

Appendix: More Proofs on Functions

Proof 1: Proving a function is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Surjective Functions

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Proof:

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Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

Proof: Consider any $y \in \mathbb{R}$. We will prove that there is a choice of $x \in \mathbb{R}$ such that $f(x) = y$.

Let $x = y / 2$.

Surjective Functions

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Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

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So we see that $f(x) = y$, as required. ■

This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Proof 2: Proving a function is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg \forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \neg \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof:

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Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$.

Surjective Functions

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Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

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Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

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Notice that $g(m) = 2m$ is even, while 137 is odd.

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